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S_p-Separation Axioms

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Abstract— In this paper S_p-open sets are used to define some new types of separation axioms in topological spaces. The implications of these separation axioms among themselves with some other separation axioms are obtained. Also their basic properties and characterizations are investigated.

Index Terms— S_p-open sets, pre separation axioms, semi-separation axioms.

1 Introduction

The notion of semi-open sets which was introduced by Levine in 1963 **[5]** is one of the well-known notion of generalized open sets. Several types of generalized open sets were introduced such as preopen sets **[7]** which was introduced by Mashhour et al in 1982. The notion of S_p-open sets **[9]** introduced by Shareef in 2007. In **[6]** Maheshwari and Prasad have defined the concept of semi-T_i, (i=0, 1, 2) spaces also in **[3]** Kar and Bhattacharyya defined new weak types of separation axioms via preopen sets called pre-T_i spaces for i=0, 1, 2 and in **[4]** Khalaf introduced strongly semi-separation axioms by using special types of semi open sets.

In this paper we define new types of separation axioms called S_p-T_i spaces which are stronger than semi-T_i spaces and weaker than strongly semi-T_i spaces (i=0,1,2).

2 Preliminaries

Throughout this paper X and Y will always denote topological spaces and $f: X \to Y$ will denote a function from a space X into a space Y. If A is a subset of , then the closure and interior of A in X are denoted by cl(A) and int(A) respectively. while $S_p$cl(A) and $S_p$int(A) denote the S_p-closure and S_p-interior of A in X respectively.

Definitions 2.1: A subset A of a space X is called:

1. semi-open **[5]**, if $A \subseteq$ cl(int(A)),
2. preopen **[7]**, if $A \subseteq$ int(cl(A)).
3. regular closed **[11]**, if A = cl(int(A)).
4. θ-semi-open **[8]**, if for each $x \in A$, there exists a semi-open set U such that $x \in U \subseteq$cl(U)$\subseteq A$.
5. S_p-open **[9],** if A is semi-open and for each $x \in A$, there exists a preclosed set F such that $x \in F \subseteq A$.

The complement of a semi-open, preopen and S_p-open set is called semi-closed, preclosed and S_p-closed set respectively.

The family of all semi-open, preopen and S_p-open sets in a space X is denoted by SO(X), PO(X) and S_pO(X) respectively, while SC(X), PC(X) and S_pC(X) denote the family of semi-closed, preclosed and S_p-closed sets in a space X respectively.

Definition 2.2: A space X is said to be:

1) Semi-T_0 **[6]**, (resp., pre-T_0 **[3]**, strongly semi-T_0 **[4]** and T_0 **[11]**) space if for each two distinct points x and y in X, there exists a semi-open (resp., preopen, θ-semi-open and open) set containing one of them but does not contain the other.
2) Semi-T_1 **[6]**, (resp., pre-T_1 **[3]**, strongly semi-T_1 **[4]** and T_1 **[11]**) space if for each two distinct points x and y in X, there exist semi-open (resp., preopen, θ-semi-open and open) sets U and V containing x and y respectively, such that $x \notin U$ and $y \notin V$.
3) semi-T_2 **[6]** (resp., pre-T_2 **[3]**, strongly semi-T_2 **[4]** and T_2 **[11]**) space if for each two distinct points x and y in X, there exist two disjoint semi-open (resp., preopen, θ-semi-open and open) sets U and V containing x and y respectively.

Definition 2.3: **[2]** A function $f: X \to Y$ is said to be s-continuous or (strongly semi-continuous) if the inverse image of each semi-open set in Y is an open set in X.

The following definitions and results are from **[9]**.

Definition 2.4: Let X be a space and let $x \in X$, then a subset N_x of X is said to be S_p-neighborhood of x if there exists S_p-open set U in X such that $x \in U \subseteq N_x$.

Lemma 2.5: If a space X is preT_1-space, then SO(X) = S_pO(X).

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Lemma 2.6: Every θ-semi-open set of X is S_p-open set.

Theorem 2.7: Let X be a space and $A, B \subseteq X$. If $A \subseteq B$, then $S_p cl(A) \subseteq S_p cl(B)$.

Lemma 2.8: The set A is S_p-open in the space X if and only if for each $x \in A$, there exists an S_p-open set B such that $x \in B \subseteq A$.

Lemma 2.9: For any subset A of a space X, $S_p cl(A) = A \cup S_p D(A)$, where $S_p D(A)$ stands for the set of all S_p-limit points of A in X.

Theorem 2.10: Let Y be a regular closed subset of X. If A is an S_p-open subset of Y, then A is S_p-open in X.

Theorem 2.11: Let $f: X \rightarrow Y$ be a homeomorphism. If $A \in S_pO(X)$, then $f(A) \in S_pO(Y)$.

Theorem 2.12: A function $f: X \rightarrow Y$ is S_p-continuous if and only if for every open subset O of Y, $f^{-1}(O)$ is S_p-open in X.

Theorem 2.13: Let $f: X \rightarrow Y$ be continuous and open function, then $f^{-1}(B) \in S_pO(X)$ for any $B \in S_pO(Y)$.

Theorem 2.14: [5] Let X_1 and X_2 be two spaces and $X = X_1 \times X_2$ be the product space. If $A \in SO(X_1)$ and $B \in SO(X_2)$, then $A \times B \in SO(X_1 \times X_2)$.

Theorem 2.15: [1] For any spaces X and Y, if $A \subseteq X$ and $B \subseteq Y$, then $pcl_{X \times Y}(A \times B) = pcl_X(A) \times pcl_Y(B)$.

3 S_p-SEPARATION AXIOMS

Definition 3.1: A space X is said to be:

1) S_p-T_o space if for each pair of distinct points in X, there exists an S_p-open set in X containing one of them and not the other.
2) S_p-T_1 space if for each pair of distinct points x and y in X, there exists two S_p-open sets U and V in X containing x and y respectively such that $y \notin U$ and $x \notin V$.
3) S_p-T_2 space if for each pair of distinct points x and y in X, there exists two disjoint S_p-open sets U and V in X such that $x \in U$ and $y \in V$.

Remark 3.2: From the above definition and Definition 2.2**,** it is clear that every S_p-T_i space is semi-T_i , for i=0, 1, 2. But the converse is not true in general as it is shown by the following examples:

Example 3.3: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$. Then $SO(X) = \tau$ and $S_pO(X) = \{\emptyset, X\}$. This implies that X is semi-T_0 space but not S_p-T_0 .

Example 3.4: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then
$SO(X)=$
$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\},$
$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$
and $S_pO(X) = \{\emptyset, X, \{a, c, d\}, \{b, c, d\}\}$. Hence, the space X is semi-T_1 but not S_p-T_1 also X is semi-T_2 but not S_p-T_2**.**

Remark 3.5: It is clear that every S_p-T_2 space is S_p-T_1 space and every S_p-T_1 space if S_p-T_0 space but the converse is not true in general as it is shown in the following examples.

Example 3.6: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Hence
$SO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, and $PC(X) = \{\emptyset, X, \{b, c\}, \{a, c\}, \{c\}\}$ also
$S_pO(X) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. Then X is S_p-T_0, but not S_p-T_1 .

Example 3.7: Let X be any infinite set equipped with the co-finite topology. Then X is T_1-space, so by Lemma 2.5, $SO(X) = S_pO(X)$ and every infinite subset of X is semi-open set. Hence X is both semi-T_1 and S_p-T_1. But it is obvious that X is not S_p-T_2 space.

Lemma 3.8: Every strongly semi-T_i space is S_p-T_i space, for i=0, 1, 2.

Proof: Let X be strongly semi-T_0 space and let $x, y \in X$ such that $x \neq y$. Then there exists a θ-semi-open set U containing one of them but not the other and since by Lemma 2.6, U is S_p-open set containing one of them but not the other. This implies that X is S_p-T_0.

Similarly we can prove for i=1 and 2.

The convers of Lemma 3.8 is not true in general as it is shown in the example below:

Example 3.9: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, then SO($X$) = $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$, θSO(X) = $\{\emptyset, X\}$ and S_pO(X) = $\{\emptyset, X, \{a, c\}, \{b, c\}\}$. Therefor, X is S_p-T_0 but not strongly semi-T_0 space.

Proposition 3.10: If a space X is S_p-T_1, then it is pre-T_1.

Proof: Let X be an S_p-T_1 space and let $x, y \in X$ such that $x \neq y$, so there exist two S_p-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. This implies that by Definition 2.1, there exist two preclosed sets F_1 *and* F_2 such that $x \in F_1 \subseteq U$ and $y \in F_2 \subseteq V$. Hence, $X \backslash F_1$ and $X \backslash F_2$ are preopen sets such that $\in X \backslash F_2$, $y \notin X \backslash F_2$ and $y \in X \backslash F_1$, $x \notin X \backslash F_1$. Therefore, by Definition 2.2, X is pre-T_1.

The converse of Proposition 3.8 is not true in general as it is seen in the example below:

Examples 3.11: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b, c\}, \{b, c, d\}, \{b, c\}\}$. Then

PO(X) = $\{\emptyset, X, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$ and S_pO(X) = $\{\emptyset, X, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. It can be checked that X is pre-T_1 but not S_p-T_1.

The property of a space being S_p-T_0 space is not hereditary property as it is shown in the following example :

Example 3.12: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, then X is S_p-T_0 space and let $Y = \{b, c\}$ and $\tau_Y = \{\emptyset, Y, \{b\}\}$, then S_pO(Y)=$\{\emptyset, Y\}$. The subspace Y is not S_p-T_0 subspace.

Proposition 3.13: The property of a space being S_p-T_i (for i=0,1,2) is a topological property.

Proof: Let $f: X \to Y$ be a homeomorphism and let X be S_p-T_0. Suppose that $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is onto so, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ and $x_1 \neq x_2$. Since X is S_p-T_0, so there exists an S_p-open set G of X containing one of the points x_1, x_2 and not the other. Since f is homeomorphism, so by Theorem 2.11, $f(G)$ is also S_p-open in Y and containing one of the points y_1, y_2 and not the other. Thus Y is also S_p-T_0 space.

The proof for the space being S_p-T_1 and S_p-T_2 is similar.

Theorem 3.14: A space X is S_p-T_0 if and only if the S_p-closure of distinct points are distinct.

Proof: Let X be S_p-T_0 and $x, y \in X$ such that $x \neq y$. Since $x \neq y$ and X is S_p-T_0, so there exists an S_p-open set G contains one of them, say x, and not the other. Then $X \backslash G$ is S_p-closed set in X contains y but not x, but $S_p$cl($\{y\}$) $\subseteq X \backslash G$ and since $x \notin X \backslash G$ implies that $x \notin S_p$cl($\{y\}$), so $S_p$cl($\{x\}$) $\neq S_p$cl($\{y\}$).

Conversely: To show that X is S_p-T_0 space, let $x, y \in X$ such that $x \neq y$. So by hypothesis, $S_p$cl($\{x\}$) $\neq S_p$cl($\{y\}$), then there exist at least one point d of X which belongs to one of them, say $S_p$cl($\{x\}$) and does not belongs to $S_p$cl($\{y\}$). If $x \in S_p$cl($\{y\}$), then $\{x\} \subseteq S_p$cl($\{y\}$). This implies that, by Theorem 2.7, $S_p$cl($\{x\}$)$\subseteq S_p$cl($\{y\}$) which is a contradiction to the fact that $d \notin S_p$cl($\{y\}$) but $d \in S_p$cl($\{x\}$), so $x \notin S_p$cl($\{y\}$). Hence, $x \in X \backslash S_p$cl($\{y\}$) and $X \backslash S_p$cl($\{y\}$) is S_p-open set containing x but not y. Thus, X is S_p-T_0 space.

Theorem 3.15: A space X is S_p-T_1 space if and only if every singleton subset of X is S_p-closed .

Proof: Let X be S_p-T_1 space and $x \in X$. Let $y \in X \backslash \{x\}$ implies that $y \neq x$ and since X is S_p-T_1 space so there exist two S_p-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. This implies that $y \in V \subseteq X \backslash \{x\}$, so by Lemma 2.8, $X \backslash \{x\}$ is an S_p-open set . Hence, $\{x\}$ is S_p-closed .

Conversely: Let $x, y \in X$ such that $x \neq y$ implies that $\{x\}, \{y\}$ are two S_p-closed sets in X. Then $X \backslash \{x\}$ and $X \backslash \{y\}$ are two S_p-open sets and $X \backslash \{x\}$ contains y but not x also $X \backslash \{y\}$ contains x but not y this implies that X is S_p-T_1 space.

Theorem 3.16: For any space X the following statements are equivalent:

1. X is S_p-T_1 space.
2. Each subset of X is the intersection of all S_p-open sets containing it.
3. The intersection of all S_p-open sets containing the point $x \in X$ is the set $\{x\}$.

Proof: (1) $\Rightarrow$ (2). Let X be S_p-T_1 and $A \subseteq X$. Then for each $y \notin A$, there exists a set $X \backslash \{y\}$ such that $A \subseteq X \backslash \{y\}$ and by Theorem 3.15, the set $X \backslash \{y\}$ is S_p-open for every y. This implies that $A = \cap \{X \backslash \{y\}: y \in X \backslash A\}$, so the intersection of all S_p-open sets containing A is A itself.

(2)$\Rightarrow$(3). Let $x \in X$, then $\{x\} \subseteq X$ so by (2), the intersection of all S_p-open sets containing $\{x\}$ is $\{x\}$ itself. Hence the intersection of all S_p-open sets containing x is $\{x\}$.

(3) $\Rightarrow$ (1). Let $x, y \in X$ such that $x \neq y$ implies that by (3), the intersection of all S_p-open sets containing x and y are $\{x\}$ and $\{y\}$ respectively, then for each $x \in X$ there exists an S_p-open set

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G_x such that $x \in G_x$ and $y \notin G_x$. Similarly for $y \in X$ there exists an S_p-open set G_y such that $y \in G_y$ and $x \notin G_y$ this implies that X is S_p-T_1 space.

Theorem 3.17: A space X is S_p-T_1 if and only if $S_pD(\{x\}) = \emptyset$ for each $x \in X$.

Proof: Let X be S_p-T_1 space and $x \in X$. If possible suppose that $S_pD(\{x\}) \neq \emptyset$ implies that there exists $y \in S_pD(\{x\})$ and $y \neq x$ and since X is S_p-T_1, so there exists an S_p-open set U in X such that $y \in U$ and $x \notin U$ implies that $\{x\} \cap U = \emptyset$, then $y \notin S_pD(\{x\})$ which is a contradiction. Thus $S_pD(\{x\}) = \emptyset$ for each $x \in X$.

Conversely: Let $S_pD(\{x\}) = \emptyset$ for each $x \in X$, then by Lemma 2.9, $S_pcl(\{x\})=\{x\}$ which is S_p-closed set in X. This implies that each singleton set in X is S_p-closed. Thus by Theorem 3.15, X is an S_p-T_1 space.

Lemma 3.18: If every finite subset of a space X is S_p-closed , then X is S_p-T_1 space.

Proof: Let $x, y \in X$ such that $x \neq y$. Then by hypothesis, $\{x\}$ and $\{y\}$ are S_p-closed sets which implies that $X\setminus\{x\}$ and $X\setminus\{y\}$ are S_p-open sets such that $x \in X\setminus\{y\}$ and $y \in X\setminus\{x\}$. Hence X is S_p-T_1 space.

Theorem 3.19: If X is S_p-T_0 space, then $S_pint(S_pcl(\{x\}))\cap S_pint(S_pcl(\{y\})) = \emptyset$ for each pair of distinct points x and y in X.

Proof: Let X be S_p-T_0 and $x, y \in X$ such that $x \neq y$. Then there exist an S_p-open set G containing one of the point, say x, and not the other implies that $x \in G$ and $y \notin G$, then $y \in X\setminus G$ and $X\setminus G$ is S_p-closed . Now $S_pint(\{y\})\subseteq S_pint(S_pcl(\{y\}))\subseteq X\setminus G$ this implies that $G \cap S_pint(S_pcl(\{y\})) = \emptyset$, then $G \subseteq X\setminus S_pint(S_pcl(\{y\}))$. But $x \in G \subseteq X\setminus S_pint(S_pcl(\{y\}))$, then $S_pcl(\{x\}) \subseteq X\setminus S_pint(S_pcl(\{y\}))$ this implies that $S_pint(S_pcl(\{x\}))\subseteq S_pcl(\{x\}) \subseteq X\setminus S_pint(S_pcl(\{y\}))$. Therefore, $S_pint(S_pcl(\{x\}))\cap S_pint(S_pcl(\{y\})) = \emptyset$.

Theorem 3.20: If for each $x \in X$, there exists a regular closed set U containing x such that U is S_p-T_0 subspace of X, then the space X is S_p-T_0.

Proof: Let x, y be two distinct points in X, then by hypothesis there exists regular closed sets U and V such that $x \in U$, $y \in V$ and U , V are S_p-T_0 subspaces. Now if $y \notin U$ then the proof is complete but if $y \in U$ and since U is S_p-T_0 subspace, so there exists an S_p-open set W in U such that $y \in W$ and $x \notin W$ and since U is regular closed set so by Theorem 2.10, W is an S_p-open set in X containing y *but not* x . Thus X is S_p-T_0 .

Similar to Theorem 3.20, we can prove the following result.

Theorem 3.21: If for each $x \in X$, there exists a regular closed set U containing x such that U is S_p-T_1 subspace of X, then the space X is S_p-T_1.

Theorem 3.22: For a space X the following statements are equivalent:

1. X is S_p-T_2 space.
2. If $x \in X$, then for each $y \neq x$ there exists an S_p-neighborhood N of x such that $y \notin S_pcl(N)$.
3. For each $x \in X$, $\cap\{S_pcl(N)$: N is S_p-neighborhood of $x\} = \{x\}$.

Proof: (1) ⇒ (2). Let X be an S_p-T_2 space and let $x \in X$, then for each $y \neq x$ there exist two disjoint S_p-open sets U and V such that $x \in U$ and $y \in V$. This implies that $x \in U \subseteq X\setminus V$, so by Definition 2.4, $X\setminus V$ is an S_p-neighborhood of x which is S_p-closed set in X and $y \notin X\setminus V$ implies that $y \notin S_pcl(X\setminus V)$.

(2) ⇒ (1). Let $x, y \in X$ such that $x \neq y$, then by hypothesis, there exists an S_p-neighborhood N of x such that $y \notin S_pcl(N)$ implies that $y \in X\setminus S_pcl(N)$ and $x \notin X\setminus S_pcl(N)$. But $X\setminus S_pcl(N)$ is S_p-open set also since N is S_p-neighborhood of x, then there exists an S_p-open set G of X such that $x \in G \subseteq N$ this implies that $G \cap (X\setminus S_pcl(N)) = \emptyset$. Hence X is S_p-T_2 .

(2) ⇒ (3). Let $x \in X$. If $\cap\{S_pcl(N)$: N is S_p-neighborhood of $x\} \neq \{x\}$, then there exists $y \in \cap\{S_pcl(N)$: N is S_p-neighborhood of $x\}$ such that $y \neq x$ so by (2), there exists an S_p-neighborhood M of x such that $y \notin S_pcl(M)$ which is contradiction to the fact that $y \in \cap\{S_pcl(N)$: N is S_p-neighborhood of $x\}$. Thus $\cap\{S_pcl(N)$: N is S_p-neighborhood of $x\} = \{x\}$.

(3) ⇒ (2). Let $x \in X$, so by hypothesis, we have $\cap\{S_pcl(N)$: N is S_p-neighborhood of $x\} = \{x\}$. Now if $\in X$ *and* $y \neq x$, then $y \notin \cap\{S_pcl(N)$: N is S_p-neighborhood of $x\} = \{x\}$ and hence there exists an S_p-neighborhood M of x such that $y \notin S_pcl(M)$.

Lemma 3.23: Let Y be a regular closed subset of the space, then any S_p-neighborhood of the point x in Y is an S_p-neighborhood of x in X .

Proof: Let N be any S_p-neighborhood of $x \in Y$ this implies that by Definition 2.4, there exists an S_p-open set G in Y such that $x \in G \subseteq N$. Since Y is regular closed set in X, so by Theorem 2.10, G is an S_p-open set in X which implies that N is an S_p-neighborhood of x in X.

Lemma 3.24: Let Y be a regular closed subspace of the space X and A $\subseteq$ Y, then S$_p$cl(A) $\subseteq$ S$_p$cl$_y$(A).

Proof: Let $x \notin$ S$_p$cl$_y$(A) implies that there exists an S$_p$-open set U in Y containing x such that U $\bigcap$ A = ϕ. Since Y is regular closed set in X then by Theorem 2.10, U is S$_p$-open set in X implies that $x \notin$ S$_p$cl(A), so S$_p$cl(A) $\subseteq$ S$_p$cl$_y$(A).

Theorem 3.25: If for each point x of a space X there exists a regular closed subset A containing x and A is S$_p$-T$_2$ subspace of X, then X is S$_p$-T$_2$ space.

Proof: Let $x \in X$, then by hypothesis, there exists a regular closed set A containing x and A is S$_p$-T$_2$ subspace. Hence, by Theorem 3.22, we have $\cap\{$S$_p$cl$_A(N)$: N is S$_p$-neighborhood of x in $A\} = \{x\}$ and since A is regular closed set in X, so by Lemma 3.24, S$_p$cl$(N) \subseteq$ S$_p$cl$_A(N)$ and by Lemma 3.23, N is S$_p$-neighborhood of x in X, so $\cap\{$S$_p$cl(N): N is S$_p$-neighborhood of x in $X\} = \{x\}$. Therefore by Theorem 3.22, X is S$_p$-T$_2$.

Theorem 3.26: A space X is S$_p$-T$_2$ if and only if for each pair of distinct points $x, y \in X$, there exists an S$_p$-clopen set U containing one of them but not the other.

Proof: Let X be S$_p$-T$_2$ space and $x, y \in X$ such that $x \neq y$ implies that there exists two disjoint S$_p$-open sets U and V such that $x \in U$ and $y \in V$. Now since $U \cap V = \emptyset$ and V is S$_p$-open set implies that $x \in U \subseteq X\backslash V$ and $X\backslash V$ is S$_p$-closed set, since X is S$_p$-T$_2$ space so for each $x \in X\backslash V$ there exists an S$_p$-open set U_x such that $x \in U_x \subseteq X\backslash V$, then by Lemma 2.8, $X\backslash V$ is S$_p$-open set. Thus $X\backslash V$ is S$_p$-clopen set.
Conversely: Let for each pair of distinct points $x, y \in X$, there exists an S$_p$-clopen set U containing x but not y implies that $X\backslash U$ is also S$_p$-open set and $y \in X\backslash U$, since $U \cap (X\backslash U) = \emptyset$ so X is S$_p$-T$_2$ space.

Theorem 3.27: A space X is S$_p$-T$_2$ space if for any pair of distinct points $x, y \in X$, there exists an S$_p$-continuous function f of X into a T$_2$-space Y such that $f(x) \neq f(y)$.

Proof: Let x and y be any two distinct points in X. Then by hypothesis there exists an S$_p$-continuous function f from X into a T$_2$-space Y such that $f(x) \neq f(y)$. But $f(x), f(y) \in Y$ and since Y is T$_2$-space so there exists two disjoint open sets U_x and V_y such that $f(x) \in U_x$ and $f(y) \in V_y$ implies that $x \in f^{-1}(U_x)$ and $y \in f^{-1}(V_y)$ and since f is S$_p$-continuous function, so by Theorem 2.12, $f^{-1}(U_x), f^{-1}(V_y)$ are S$_p$-open sets and $f^{-1}(U_x) \cap f^{-1}(V_y) = \emptyset$. This implies that X is S$_p$-T$_2$ space.

Theorem 3.28: For a space X the following statements are equivalent:

1. X is S$_p$-T$_2$ space.
2. The intersection of all S$_p$-clopen sets of each point in X is singleton.
3. For a finite number of distinct points x_i ($1 \leq i \leq n$), there exists an S$_p$-open set G_i such that G_i ($1 \leq i \leq n$) are pairwise disjoint.

Proof:
(1) ⇒(2). Let X be S$_p$-T$_2$ space and $x \in X$. To show $\cap\{G: G$ is S$_p$-clopen and $x \in G\} = \{x\}$. If $\cap\{G: G$ is S$_p$-clopen and $x \in G\} = \{x, y\}$ where $x \neq y$. Then since X is S$_p$-T$_2$ space so there exists two disjoint S$_p$-open sets U and V such that $x \in U$ and $y \in V$, implies that $x \in U \subseteq X\backslash V$ so by Lemma 2.8, $X\backslash V$ is S$_p$-open set and also it is S$_p$-closed set this implies that $X\backslash V$ is S$_p$-clopen containing x but not y which is a contradiction. Thus the intersection of all S$_p$-clopen sets containing x is $\{x\}$.
(2) ⇒ (3). Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a finite number of distinct points of X, then by (2), $\{x_i\} = \cap\{F: F$ is S$_p$-clopen set and $x_i \in F\}$ for $i = 1, 2, \ldots, n$. Since $x_i \notin \{x_j\}$, for $i, j = 1, 2, \ldots, n$ and $i \neq j$, so there exists an S$_p$-clopen set F_0 such that $x_i \in F_0$ and $x_j \notin F_0$ for $i \neq j$, ($1 \leq i, j \leq n$) implies that $x_j \in X\backslash F_0$, where $X\backslash F_0$ is also S$_p$-clopen set and $F_0 \cap (X\backslash F_0) = \emptyset$. Therefore $X\backslash F_0$ is S$_p$-open set containing x_j, that is for each i there exist pairwise disjoint S$_p$-open sets N_i for x_i ($1 \leq i \leq n$).
(3) ⇒ (1). Obvious

Lemma 3.29: Let X_1 and X_2 be two spaces and $X_1 \times X_2$ be a product space. If $A_1 \in$ S$_p$O(X_1) and $A_2 \in$ S$_p$O(X_2), then $A_1 \times A_2 \in$ S$_p$O$(X_1 \times X_2)$.

Proof: Let $A_1 \in$ S$_p$O(X_1) and $A_2 \in$ S$_p$O(X_2) implies that $A_1 \in$ SO(X_1) and $A_2 \in$ SO(X_2), then by Theorem 2.14, $A_1 \times A_2 \in$ SO$(X_1 \times X_2)$. And now let $(a_1, a_2) \in A_1 \times A_2$, then $a_1 \in A_1$ and $a_2 \in A_2$, but $A_1 \in$ S$_p$O(X_1) and $A_2 \in$ S$_p$O(X_2) so there exists preclosed sets $F_1 \in$ PC(X_1) and $F_2 \in$ PC(X_2) such that $a_1 \in F_1 \subseteq A_1$ and $a_2 \in F_2 \subseteq A_2$ implies that $(a_1, a_2) \in F_1 \times F_2 \subseteq A_1 \times A_2$ and $F_1 \times F_2$ is preclosed set in the product space $X_1 \times X_2$ because by Theorem 2.15, $F_1 \times F_2 = pcl_{X_1}(F_1) \times pcl_{X_2}(F_2) = pcl_{X_1 \times X_2}(F_1 \times F_2)$. Thus $A_1 \times A_2 \in$ S$_p$O$(X_1 \times X_2)$.

Theorem 3.30: Let $\{X_i : i = 1, 2, \ldots, n\}$ be any finite family of spaces. If X_i is an S_p-T_2 space for each $i = 1, 2, \dots, n$, then the product space $\prod_{i=1}^{n} X_i$ is S_p-T_2 .

Proof: Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be any two distinct points in $\prod_{i=1}^{n} X_i$, then $x_i \neq y_i$ for some $i = 1, 2, \dots, n$. Suppose that $x_1 \neq y_1$ and since X_i is S_p-T_2 space for each $i = 1, 2, \dots, n$, so there exist two disjoint S_p-open sets G and H in X_1 such that $x_1 \in G$ and $y_1 \in H$. Then by Lemma 3.29, $G \times \prod_{i=1}^{n} X_i$ and $H \times \prod_{i=1}^{n} X_i$ are S_p-open sets in $\prod_{i=1}^{n} X_i$ such that $x \in G \times \prod_{i=1}^{n} X_i$, $y \in H \times \prod_{i=1}^{n} X_i$ and $(G \times \prod_{i=1}^{n} X_i) \cap (H \times \prod_{i=1}^{n} X_i)$ = $(G \cap H) \times \prod_{i=1}^{n} X_i = \emptyset$. Hence $\prod_{i=1}^{n} X_i$ is S_p-T_2 .

Theorem 3.31: Let $f: X \to Y$ be an open continuous function. If Y is an S_p-T_2 space, then the set $\{(x_1, x_2): f(x_1) = f(x_2)\}$ is an S_p-closed set in the product space $X \times X$.

Proof: Let $A = \{(x_1, x_2): f(x_1) = f(x_2)\}$. It is enough to show $(X \times X)\backslash A$ is an S_p-open set, so let $(a_1, a_2) \in (X \times X)\backslash A$, then $f(a_1) \neq f(a_2)$. But $f(a_1), f(a_2) \in Y$ and Y is S_p-T_2 space, so there exist two disjoint S_p-open sets U and V such that $f(a_1) \in U$ and $f(a_2) \in V$ implies that $a_1 \in f^{-1}(U)$ and $a_2 \in f^{-1}(V)$ and since f is open and continuous function, so by Theorem 2.13, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint S_p-open sets in X, then by Lemma 3.29, $f^{-1}(U) \times f^{-1}(V)$ is an S_p-open set in $X \times X$. Hence, $(a_1, a_2) \in f^{-1}(U) \times f^{-1}(V) \subseteq (X \times X)\backslash A$ and therefore by Lemma 2.8, $(X \times X)\backslash A$ is an S_p-open set in $X \times X$. This implies that A is an S_p-closed set in $X \times X$.

Theorem 3.32: If f and g are s-continuous (or strongly semi-continuous) functions on a space X into an S_p-T_2 space Y, then the set of all point x in X such that $f(x) = g(x)$ is closed set in X.

Proof: Let $A = \{x \in X: f(x) = g(x)\}$. It is enough to show that $X\backslash A$ is an open set in X. So let $a \in X\backslash A$, then $f(a) \neq g(a)$ and $f(a), g(a) \in Y$, but Y is S_p-T_2 space, hence, there exist two disjoint S_p-open sets U and V in Y such that $f(a) \in U$ and $g(a) \in V$. Since f and g are s-continuous functions and U, V are semi-open sets, so by Definition 2.3, we obtain that $f^{-1}(U)$ and $g^{-1}(V)$ are open sets containing a. This implies that $a \in f^{-1}(U) \cap g^{-1}(V)$ and $f^{-1}(U) \cap g^{-1}(V)$ is open set also. Now let $E = f^{-1}(U) \cap g^{-1}(V)$ then we must show that $E \subseteq X\backslash A$. If possible, suppose that there exists one point $b \in E$ but $b \notin X\backslash A$, then $b \in A$. Therefore, $f(b) = g(b)$ and since $b \in E$, then $b \in f^{-1}(U)$ and $b \in g^{-1}(V)$. This implies that $f(b) \in U$ and $g(b) \in V$, but $f(b) = g(b)$ so $U \cap V \neq \emptyset$ which is contradiction. Thus $a \in E \subseteq X\backslash A$ implies that $X\backslash A$ is a neighborhood of each of it's points, so $X\backslash A$ is open set. Thus A is closed set in X.

Corollary 3.33: If f and g are s-continuous (or strongly semi-continuous) functions on a space X into an S_p-T_2 space Y and the set of all points x in X such that $f(x) = g(x)$ is dense in X, then $f = g$.

Proof: By Theorem 3.32 , we have the set $A = \{x \in X: f(x) = g(x)\}$ is closed in X, that is A = cl(A) and from the hypothesis A is dense implies that A = cl(A) = X. Therefore, $f(x) = g(x)$ for all $x \in X$. Hence $f = g$.

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